**Mu'tah University**

**College of Graduate Studies**

# Laplace Transform Applications of Fractional Differential Equations

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## Laplace transform application of fractional differential equations

استكمالاً لمتطلبات الحصول على درجة الماجستير في **الفيزياء.**

**القسم: الفيزياء.**

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## DEDICATION

I would like to dedicate this work to my parents spirits, husband, brothers and to all my family members and friends.

## ACKNOWLEDGEMENTS

In the name of Allah. The beneficent, the merciful, I begin.

I am grateful to almighty Allah for the help and blessing that have covered me in making this research to complete my masters degree. I am also grateful to my supervisor **prof. Dr. Khaled Nawafleh,** for his guidance and encouragement throughout all stages of this work.

For my family, specially my husband who supported me in all the steps to get Master degree, this is for you all.

**Eman Mohammad Al-qatawneh**

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# ABSTRACT

## Laplace Transform Applications of Fractional Differential Equations

**Eman Mohammad Al-qatawneh**

**Mu′tah University, 2017**

The aim of this study is to find a solution for some applications of fractional differential equations in physics using the Laplace transform technique.

In this work, after the fractional Laplace transform was reviewed, we found the solution for some of fractional differential equations including conservative systems such as, harmonic oscillator equation, and for non- conservative systems such as RL, and RLC circuits.

الملخص

تطبيقات تحويل لابلاس للمعادلات التفاضلية الكسرية

إيمان محمد القطاونة

جامعة مؤتة، 2017

الهدف من هذه الدراسة هو إيجاد حل لمجموعة من المعادلات التفاضلية الكسرية لبعض التطبيقات في الفيزياء باستخدام تقنية تحويل لابلاس.

في هذا العمل، وبعد مراجعة الكسور الجزئية لتحويلات لابلاس ، تم إيجاد الحل لبعض من المعادلات التفاضلية الكسرية التي تتضمن الأنظمة المحافظة، مثل معادلة الهزاز التوافقي، والأنظمة غير المحافظة، مثل دائرة مقاومة ومحث ودائرة مقاومة ومحث ومواسع.

# CHAPTR ONE

# INTRODUCTION

## 1-1 Statement of the Problem

The aim of this study is to use the fractional Laplace transform definition for solving fractional differential equations including conservative and non-conservative systems.

## 1-2 Motivation

Fractional calculus is one of the generalizations of classical calculus and has been used in different fields of science and engineering. A large amount of mathematical knowledge on fractional integrals and derivatives has been improved (Agrawal, (2001)).

Fractional calculus is the branch which generalizes the derivative of a function to non-integer order, allowing calculations such as deriving a function to $1/2$ order (Xuru, (2006)).

This study seeks to use fractional Laplace transform for the fractional differential equations that appear in mathematics, physics, and engineering. Such as solving some of fractional differential equations for conservative systems, and for non- conservative systems such as RL circuit, and RLC circuit.

## 1-3 Previous Studies

Fractional calculus was born in 1695, when L'Hopital asked Leibniz in his letter about particular notation he had used in his brochure for the nth-derivative of the linear function $f(x) = \frac{d^n x}{dx^n}$. L'Hopital posed the question to Lebniz, "What would the result be if the order will be $\mathrm{n} = 1/2$. Leibniz responded "it will lead to a paradox, from which one day useful consequences will be drawn". From these words fractional calculus was born (Adam Loverro (2004)).

After that, fractional calculus was studied from best minds in mathematics, Fourier, Euler, and Laplace. Several kinds of definitions for fractional calculus constructed, Riemann, Able, Grunwald-Letenkove, and others (Adam Loverro (2004)).

Arfken and Weber (1985), introduced the Laplace transform for converting differential equations into simpler forms that may be solved more easily.

S. Z. Rida and A. A. M. Arafa (2011), developed a new application of the Mittag-Leffler Function method that will extend the application of the method to linear differential equations with fractional order.

Eltayeb. A. M. Yousif and Fatima. A. Alawad (2012), applied the Laplace transform method for solving the fractional ordinary differential equations with constant and variable coefficients.

Saeed kazem (2013), applied the Laplace transform for solving linear fractional-order differential equation. It will allow us to transform fractional differential equations into algebraic equations and then by solving this algebraic equation, We can obtain the unknown function by using the inverse Laplace Transform.

Shy-Der Lin and Chia-Hung Lu (2013), presented how this simple fractional calculus method to the solutions of some families of fractional differential equations would lead naturally to several interesting consequences.

Ali M Qudah (2013), used the fractional calculus methods to solve essential problems in conservative and non-conservative oscillatory systems.

Danny Vance (2014), provided a basic introduction to fractional calculus, a branch of mathematical analysis that studies the possibility of taking any real power of the differentiation operator.

Moaz M Altarawneh (2015), proposed fractional differential equations for the motion of a charged particle in a uniform electric field at right angle with a uniform magnetic field (cycloid motion), an exact general solution interms of the Mittag-Leffler function was obtained for the fractional differential equations by employing the Laplace transform technique.

**1-4 Outline of Thesis**

This thesis is organized in four chapters. Chapter one is the introductory chapter, which merely states and motivates the problem. In chapter two, we talked about the fractional Laplace transform, and we described the history of fractional calculus, showed the properties of the fractional derivative. Chapter three presents the classical field treatment of solving some of differential equations such as the harmonic oscillator equation, RL circuit, RLC circuit equations by using Laplace transform. In chapter four, the fractional treatment is introduced using the fractional Laplace transform . Finally, some concluding remarks are given.

## CHAPTER TWO
## FRACTIONAL CALCULUS

### 2-1 History of Fractional Calculus

The idea of fractional calculus began in 1695 when the scientist L'Hopital asked Leibniz in his letter about particular notation he had used in his brochure for the nth-derivative of the linear function $f(x) = \frac{d^n x}{dx^n}$. L'Hopital posed the question to Lebniz, "What would the result be if the order will be n $= 1/2$. Leibniz responsed "it will lead to a paradox, from which one day useful consequences will be drawn". From these words fractional calculus was born (Adam Loverro (2004)), ( Dalir and Bashour (2010)).

### 2-2 Applications of Fractional Calculus

Fractional order calculus can represent systems with high-order dynamics and complex nonlinear phenomena. Another important property is that fractional order derivatives depend not only on local conditions of the evaluated time but also on the entire history of the function. (S.A. David, J.L. Linares e E.M.J.A. Pallone, (2011)).

There are many applications of Fractional calculus such as in fluid mechanics which use Fractional calculus for solution of time-dependent, viscous–diffusion fluid mechanics problems, electric transmission lines, and many other applications (Dalir and Bashour,(2010)).

### 2-3 Useful Mathematical Functions In Fractional Calculus

There are many useful Mathematical definitions In Fractional Calculus, we will discuss some of them such as the Gamma function, the Beta function, the Mittag-Leffler function (Arfken and Weber (1985), Joseph M. kimeu, (2009)).

### 2-3-1 The Gamma Function

The most basic interpretation of the Gamma function is simply the generalization of the factorial for all real numbers. Its definition is given by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt, \qquad x \in \mathbb{R}^+ \tag{2.1}$$

The Gamma function has some special properties. By using its recursion relations we can obtain formulas

$$\Gamma(x+1) = x\Gamma(\mathrm{x}), \qquad \mathrm{x} \in \mathbb{R}^+ \tag{2.2}$$

$$\Gamma(x) = (x-1)!, \qquad x \in \mathbb{N} \tag{2.3}$$

(Arfken and Weber (1985)).

For example, let us evaluate $\Gamma(1/2)$

From equation (2.3) we note that $\Gamma(1) = 1$. By definition equation (2.1) we have

$$\Gamma(1/2) = \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt$$

If we let $t = y^2$, then $dt = 2\,y\,dy$, and we now have

$$\Gamma(1/2) = 2\int_0^\infty e^{-y^2} dy \tag{2.4}$$

Equivalently, we can write equation (2.4) as

$$\Gamma(1/2) = 2\int_0^\infty e^{-x^2} dx \tag{2.5}$$

If we multiply together equation (2.4) and equation (2.5) we get

$$[\Gamma(1/2)]^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dxdy \tag{2.6}$$

Equation (2.6) is a double integral over the first quadrant, and can be evaluated in polar coordinates to get

$$[\Gamma(1/2)]^2 = 4\int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta = \pi \tag{2.7}$$

Thus, $\Gamma(1/2) = \sqrt{\pi}$. (Arfken and Weber (1985)).

### 2-3-2 The Beta Function

Like the Gamma function, the Beta function is defined by a definite integral. Its definition is given by

$$\beta(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \qquad x, y \in \mathbb{R}^+ \tag{2.8}$$

The Beta function can also be defined in terms of the Gamma function as

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad , \quad x,y \in \mathbb{R}^+ \tag{2.9}$$

For example, let us evaluate $\beta(1,2)$, if x =1, y =2
From equation (2.9)

$$\beta(\mathrm{x},\mathrm{y}) = \frac{\Gamma(\mathrm{x})\,\Gamma(\mathrm{y})}{\Gamma(\mathrm{x}+\mathrm{y})}$$

$$\beta\,(1,2) = \frac{\Gamma(1)\,\Gamma(2)}{\Gamma(1+2)} = \frac{\Gamma(1)\,\Gamma(2)}{\Gamma(3)}$$

We note that $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2$

So we can write $\beta$ (1,2) = (1) (1) / (2) =1/2. (Arfken and Weber (1985)).

**2-3-3 The Mittag-Leffler Function**

The Mittag-Leffler function plays an essential role in the solution of fractional order differential equations. There are many applications of Mittag-Leffler function such as, fluid flow, electric networks, probability, statistical distrubition theory (Banu Yilmaz Yasar (2014)).

The Mittag-Leffler function is a direct generalization of the exponential function, $e^x$, and it plays an important role in fractional calculus. The one and two-parameter representations of the Mittag-Leffler function can be defined in terms of a power series as

$$\mathrm{E}_{\propto}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\propto \mathrm{k}+1)}, \quad \propto > 0 \tag{2.10}$$

$$\mathrm{E}_{\propto,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\propto \mathrm{k}+\beta)}, \quad \propto > 0, \beta > 0 \tag{2.11}$$

(Adam Loverro (2004), Joseph M. kimeu, (2009), S. Z. Rida and A. A. M. Arafa (2011)).

The exponential series defined by equation (2.11) gives a generalization of equation (2.10), this more generalized form was introduced by R.P. Agarwal in 1953.

As a result of the last definition in equation (2.11), the following relations can be written as

$$\mathrm{E}_{\propto,\beta}(x) = \frac{1}{\Gamma(\beta)} + x\,\mathrm{E}_{\propto,\propto+\beta}(x) \tag{2.12}$$

And

$$\mathrm{E}_{\propto,\beta}(x) = \beta\, \mathrm{E}_{\propto,\beta+1}(x) + \propto x \frac{d}{dx} \mathrm{E}_{\propto,\beta+1}(x) \tag{2.13}$$

Equation (2.13) implies that

$$\frac{d}{dx} \mathrm{E}_{\propto,\beta+1}(x) = \frac{1}{\propto x}\left[E_{\propto,\beta}(x) - \beta\, \mathrm{E}_{\propto,\beta+1}(x)\right].$$

So

$$\frac{d}{dx} \mathrm{E}_{\propto,\beta}(x) = \frac{1}{\propto x}\left[E_{\ \propto,\beta-1}(x) - (\beta - 1)\, \mathrm{E}_{\propto,\beta}(x)\right]. \tag{2.14}$$

Now we will prove equation (2.12). Using equation (2.11), we get

$$E_{\ \propto,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\propto \mathrm{k} + \beta)}$$

$$= \sum_{k=-1}^{\infty} \frac{x^{k+1}}{\Gamma[\propto (\mathrm{k}+1) + \beta]}$$

$$= \sum_{k=-1}^{\infty} \frac{x x^k}{\Gamma[\propto \mathrm{k} + (\propto + \beta)]}$$

$$= \frac{1}{\Gamma(\beta)} + x \sum_{k=0}^{\infty} \frac{x^k}{\Gamma[\propto \mathrm{k} + (\propto + \beta)]}$$

$$= \frac{1}{\Gamma(\beta)} + x \mathrm{E}_{\propto,\propto+\beta}(x)$$

Also, for some specific values of $\propto$ and $\beta$, the Mittag-Leffler function reduces to some popular functions. For example,

$$\mathrm{E}_{1,1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\mathrm{k}+1)} = \sum_{k=0}^{\infty} \frac{x^k}{\mathrm{k}!} = e^x \tag{2.15}$$

$$E_{1,2}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+2)} = \frac{1}{x}\sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!} = \frac{e^x - 1}{x} \tag{2.16}$$

( Eltayeb. A. M. Yousif and Fatima. A. Alawad (2012)).
There are other special cases of the Mittag-Leffler function such as

(i) $E_\alpha(x) = \frac{1}{1-x}$ , $|x| < 1$

(ii) $E_1(x) = e^x$

(iii) $E_2(x) = \cosh(\sqrt{x})$, $\quad x \in \mathbb{C}$

Where $\mathbb{C}$ being the set of complex numbers ( Deshna Loonker and P. k. Banerji (2016)).

**2-4 Fractional Derivatives**

We can define fractional differentiation of any positive real power by combining the standard differential operator with a fractional integral of order between 0 and 1. We have to choose which operator to apply first. For example, we can define the derivative of order 1.5 of a function f(t) as either of the following:

$$D^{1.5}f(t) = D^2 J^{.5} f(t)$$

$$D^{1.5}f(t) = J^{.5} D^2 f(t)$$

These two approaches supply the basis for two different definitions of the fractional derivative. The first definition, in which the fractional integral is applied before differentiating, is called the Riemann-Liouville fractional derivative. The second, in which the fractional integral is applied afterwards, is called the Caputo derivative. These two forms of the fractional derivative each behave differently. Here are their formal definitions:

**Definition 1** Pick some $\alpha \in \mathbb{R}^+$, let n be the nearest integer greater than $\alpha$. The Riemann-Liouville fractional derivative of order $\alpha$ of a function f(t) is given by:

$$D^{\alpha}f(t) = \frac{d^n}{dt^n} J^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-u)^{n-\alpha-1} f(u) du \quad (2.17)$$

**Definition 2** Pick some $\alpha \epsilon \mathbb{R}^+$, let n be the nearest integer greater than $\alpha$. The Caputo fractional derivative of order $\alpha$ of a function f(t) is given by:

$$^{c}D^{\alpha}f(t) = J^{n-\alpha} \frac{d^n}{dt^n} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-u)^{n-\alpha-1} f^{(n)}(u) du \qquad (2.18)$$

For $\alpha$<0, we use the definition $D^{\alpha}f(t) = {}^{c}D^{\alpha}f(t) = J^{-\alpha}f(t)$. For $\alpha = 0$, we set $D^0 f(t) = {}^{c}D^0 f(t) = f(t)$.
(Danny Vance, (2014)).

**2-4-1 Left-Right Fractional Derivative**

The derivative of an arbitrary real order $\alpha$ can be considered as ${}_aD_t^{\alpha}f(t)$ where the subscripts a and t denote the low limits related to the operation of fractional differentiation. Assume that the function f(t) is defined in the interval $[a,b]$.

The fractional derivative with lower terminal at the left end of the interval $[a,b]$, ${}_aD_t^{\alpha}f(t)$ is called left fractional derivative. The fractional derivative with upper terminal of the right end of the interval $[a,b]$, ${}_tD_b^{\alpha}f(t)$ is called right fractional derivative (podlubny, 1999).

**2-4-2 Left-Right Riemann-Liouville Fractional Derivative**

The left- Riemann-Liouville fractional derivative, which is denoted by LRLFD, reads as (Podlubny,1999), (Agrawal 2001).

$${}_aD_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau \qquad (2.19)$$

And the form of right Riemann-Liouville fractional derivative, which is denoted by RRLFD, is given as:

$${}_tD_b^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau) d\tau \qquad (2.20)$$

Here $\alpha$ is the order of derivative such that n-1< $\alpha \leqslant$ n and is not equal zero. If $\alpha$ is an integer, these derivatives become the usual derivatives.

$$ {}_aD_t^\alpha f(t) = \left(\frac{d}{dt}\right)^\alpha f(t) $$

$$ {}_tD_b^\alpha f(t) = \left(-\frac{d}{dt}\right)^\alpha f(t) $$

(Agrawal 2001).

**2-4-3 Left-Right Caputo Fractional Derivative**

The left Caputo Fractional Derivative which is denoted by LCFD reads as

$$ {}_a^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n f(\tau) d\tau \qquad (2.21) $$

And right Caputo Fractional Derivative which is denoted by RCFD reads as

$$ {}_t^CD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} \left(-\frac{d}{d\tau}\right)^n f(\tau) d\tau \qquad (2.22) $$

(Miller and Ross (1993)).

**2-4-4 Properties Of Fractional Derivative**

**1. Linearity**

The fractional differentiation is a linear operator

$$ D^n\big(\lambda f(t) + \mu g(t)\big) = \lambda D^n f(t) + \mu D^n g(t) $$

(Miller and Ross (1993)).

**2. Product rule**

The classical product rule for Riemann-Liouville derivative is for all $\alpha > 0$

$$\int_a^b {}_aD_t^{\alpha} f(t)g(t)dt = \int_a^b f(t)\ {}_tD_b^{\alpha}\ g(t)dt$$

**3. Composition with Fractional Derivative**

Consider the fractional derivative of order $q$ of fractional derivative of order $p$ then

$${}_aD_t^{q}\left({}_aD_t^{p} f(t)\right) = {}_aD_t^{p}\left({}_aD_t^{q} f(t)\right) = {}_aD_t^{p+q} f(t)$$

(Miller and Ross(1993).

Now let us present an example about using fractional derivatives for any function such as:

Let f(t) = 6 we want to find ${}_aD_t^{\alpha} f(t)$ in terms of RL Fractional derivative. Substituting f(t) in to equation (2.19)

$${}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau$$

We get

$${}_aD_t^{\alpha} 6 = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1}\, 6\, d\tau \tag{2.23}$$

Take $\alpha = \frac{1}{2}$ and n =1

So

$${}_aD_t^{\frac{1}{2}} 6 = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{d}{dt}\right)^1 \int_a^t (t-\tau)^{\frac{-1}{2}}\, 6\, d\tau$$

Let $\tau = ut,\ then\ d\tau = tdu,$ therefore

$${}_0D_t^{\frac{1}{2}} 6 = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{d}{dt}\right)^1 \int_0^1 (t-ut)^{\frac{-1}{2}}\, 6\, t\, du$$

$$ {}_0D_t^{\frac{1}{2}}6 = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{d}{dt}\right)^1 6\, t^{\frac{1}{2}} \int_0^1 (1-u)^{\frac{-1}{2}}\, du $$

Using the Beta Function, which is defined in equation (2.8) (Joseph M. kimeu,(2009)).

$$ \beta(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1} dt $$

And in terms of Gamma Function, it can be written as

$$ \beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} $$

Then

$$ {}_0D_t^{\frac{1}{2}}6 = \frac{1}{2\Gamma\left(\frac{1}{2}\right)}\, t^{\frac{-1}{2}}\, \beta\left(\frac{1}{2},1\right) $$

Or

$$ {}_0D_t^{\frac{1}{2}}6 = \frac{6}{2\Gamma\left(\frac{3}{2}\right)}\, t^{\frac{-1}{2}} = \frac{3}{\Gamma\left(\frac{3}{2}\right)}\, t^{\frac{-1}{2}} $$

(Maria Kamenova Ishteva ( 2005)).

**2-5 Fractional Laplace Transform**

**2-5-1 Laplace Transform**

The Laplace transform f (s) or $\mathcal{L}$ of a function F(t) is defined as

$$ f(s) = \mathcal{L}\{F(t)\} = \lim_{a\to\infty} \int_0^a e^{-st}\, F(t)dt = \int_0^\infty e^{-st}\, F(t)dt \qquad (2.24) $$

(Shy-Der Lin and Chia-Hung Lu(2013)).

To introduce the Laplace transform, let us apply the operation to some of the elementary functions:

1- Let $F(t) = e^{-kt}$

Applying Laplace transform definition,

$$\mathcal{L}\{e^{-kt}\} = \int_0^\infty e^{-k}\, e^{-s}\, dt = \int_0^\infty e^{-(s+k)t}\, dt$$

By making the integration, we get

$$f(s) = \frac{-1}{s+k}\left[e^{-\infty} - e^{0}\right]$$

We know that $e^0$=1 and $e^{-\infty} = 0$, so

$$f(s) = \frac{1}{s+k}$$

2- Let $F(t) = \sin kt$

Applying Laplace transform,

$$\mathcal{L}\{\sin kt\} = \int_0^\infty \sin kt\, e^{-st} dt$$

We can use the relation,

$$\sin kt = \frac{e^{ikt} - e^{-ikt}}{2i}$$

Then we get,

$$\mathcal{L}\{\sin kt\} = \frac{1}{2i}\int_0^\infty \left(e^{ikt} - e^{-ikt}\right) e^{-s}\, dt$$

$$\mathcal{L}\{\sin kt\} \quad = \frac{1}{2i}\int_0^\infty e^{ikt-st}\, dt - \frac{1}{2i}\int_0^\infty e^{-ikt-st}\, dt$$

$$f(s) = \frac{1}{2i}\left[\frac{1}{ik-s} e^{ikt-st}\Big|_0^\infty - \frac{1}{-ik-s} e^{-ikt-st}\Big|_0^\infty\right]$$

$$f(s) = \frac{1}{2}\left[\frac{1}{k+is} + \frac{1}{k-is}\right]$$

$$f(s) = \frac{k}{s^2 + k^2}$$ (**Arfken and Weber** (1985)).

**2-5-2 Inverse Transforms**

There is little importance to these operations unless we can carry out the inverse transform, as in Fourier transforms. That is, with

$$\mathcal{L}\{F(t)\} = f(s)$$

Then

$$\mathcal{L}^{-1}\{f(s)\} = F(t) \quad (2.25)$$

( Saeed Kazem(2013)).

The inverse transform can be determined in different ways. A table of transforms can be built up and used to carry out the inverse transformation and use a general technique.(Arfken and Weber (1985)).

**Example :** (Arfken and Weber (1985))

$$\mathbf{f(s)} = \frac{\mathbf{s}}{(\mathbf{s}+\mathbf{1})^2 + \mathbf{4}}$$

we can write the previous equation as

$$\mathbf{f(s)} = \frac{\mathbf{s+1-1}}{(\mathbf{s}+\mathbf{1})^2 + \mathbf{4}} = \frac{\mathbf{s+1}}{(\mathbf{s}+\mathbf{1})^2 + \mathbf{4}} - \frac{\mathbf{1}}{(\mathbf{s}+\mathbf{1})^2 + \mathbf{4}}$$

which can be written also as

$$\mathbf{f(s)} = \frac{\mathbf{s+1}}{(\mathbf{s}+\mathbf{1})^2 + \mathbf{4}} - \frac{\mathbf{1}}{\mathbf{2}}\frac{\mathbf{2}}{(\mathbf{s}+\mathbf{1})^2 + \mathbf{4}}$$

Take the inverse transform for both sides,

$$\boldsymbol{f(t) = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}}$$

From the table of inverse transform (Table 2.1), we have

$$\boldsymbol{f(t) = e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t}$$

**Table 2.1 inverse transform (Arfken and Weber (1985)).**

| f(s) | F(t) | Limitation |
|---|---|---|
| 1. 1 | $\delta(t)$ | *Singularity at* $+0$ |
| 2. $\frac{1}{s}$ | 1 | $s > 0$ |
| 3. $\frac{n!}{s^{n+1}}$ | $t^n$ | $n > -1$ |
| 4. $\frac{1}{s-k}$ | $e^{kt}$ | $s > k$ |
| 5. $\frac{1}{(s-k)^2}$ | $te^{kt}$ | $s > k$ |
| 6. $\frac{s}{s^2-k^2}$ | $\cosh kt$ | $s > k$ |
| 7. $\frac{k}{s^2-k^2}$ | $\sinh kt$ | $s > k$ |
| 8. $\frac{s}{s^2+k^2}$ | $\cos kt$ | $s > 0$ |
| 9. $\frac{k}{s^2+k^2}$ | $\sin kt$ | $s > 0$ |
| 10. $\frac{s-a}{(s-a)^2+k^2}$ | $e^{at}\cos kt$ | $s > a$ |
| 11. $\frac{k}{(s-a)^2+k^2}$ | $e^{at}\sin kt$ | $s > a$ |
| 12. $\frac{s^2-k^2}{(s^2+k^2)^2}$ | $t\ \cos kt$ | $s > 0$ |
| 13. $\frac{2ks}{(s^2+k^2)^2}$ | $t\ \sin kt$ | $s > 0$ |
| 14. $(s^2+k^2)^{-1/2}$ | $J_0(at)$ | $s > 0$ |

| | | |
|---|---|---|
| 15. $(s^2 - k^2)^{-1/2}$ | $I_0(at)$ | $s > a$ |
| 16. $\frac{1}{a}\cot^{-1}\left(\frac{s}{a}\right)$ | $j_0(at)$ | $s > 0$ |
| 17. $\left.\begin{matrix} \frac{1}{2a} ln \frac{s+a}{s-a} \\ \frac{1}{a} coth^{-1}\left(\frac{s}{a}\right) \end{matrix}\right\}$ | $i_0(at)$ | $s > 0$ |
| 18. $\frac{(s-a)^n}{s^{n+1}}$ | $L_n(at)$ | $s > 0$ |
| 19. $\frac{1}{s}\ln(s+1)$ | $E_1(x) = -Ei(-x)$ | $s > 0$ |
| 20. $\frac{\ln s}{s}$ | $-\ln t - \gamma$ | $s > 0$ |
| 21. $\frac{k}{(s-a)^2 - k^2}$ | $e^{at} \sinh kt$ | $s > a$ |

### 2-5-3 Laplace Transform of Derivatives

Despite that the main application of Laplace transforms is in converting differential equations into simpler forms that may be solved more easily. It will be seen, that coupled differential equations with constant coefficients transform to simultaneous linear algebraic equations. (Arfken and Weber (1985)).

The transform of the first derivative of F(t):

$$\mathcal{L}\{\dot{F}(t)\} = \int_0^{\infty} e^{-st} \frac{dF(t)}{dt} dt$$

Integrating by parts, we obtain

$$\mathcal{L}\{\dot{F}(t)\} = e^{-st}F(t)\,\Big|_{0}^{\infty} + \mathbf{s}\int_{0}^{\infty} e^{-st}\,F(t)dt$$

$$= s\mathcal{L}\{F(t)\} - F(0) \qquad (2.26)$$

Strictly speaking, F(0) = F(+0) and dF/dt is required to be at least piecewise continuous for $0 \leq t < \infty$. Naturally, both F(t) and its derivative must be the integrals do not diverge.
An extension gives

$$\mathcal{L}\{F^{(2)}(t)\} = s^2\mathcal{L}\{F(t)\} - sF(+0) - \dot{F}(+0) \qquad (2.27)$$

$$\mathcal{L}\{F^{(n)}(t)\} = s^n\mathcal{L}\{F(t)\} - s^{n-1}F(+0) - \cdots - F^{(n-1)}(+0). \qquad (2.28)$$

(Arfken and Weber (1985))

**Example**
Use Laplace transform to solve the following differential equation

$$\dot{x}(t) + 3x(t) = e^{2t}, x(0) = -1$$

(Pavel Pyrih (2012))

**Solution:**
Applying Laplace transform we get

$$\mathcal{L}(\dot{x}(t) + 3x(t)) = \mathcal{L}(e^{2t})$$

$$\mathcal{L}(\dot{x}(t)) + 3\,\mathcal{L}(x(t)) = \mathcal{L}(e^{2t})$$

$$sX(s) - X(0) + 3X(s) = \frac{1}{s-2}$$

By taking the common factor X(s) and $x(0) = -1$ we get

$$X(s)(s+3) + 1 = \frac{1}{s-2}$$

$$(s+3)X(s) = \frac{1}{s-2} - 1 = \frac{1-s+2}{s-2} = \frac{3-s}{s-2}$$

$$X(s) = \frac{3 - s}{(s - 2)(s + 3)}$$

By using partial fraction method we get

$$X(s) = \frac{3 - s}{(s - 2)(s + 3)} = \frac{A}{(s - 2)} + \frac{B}{(s + 3)}$$

Now

$$3 - s = A(s + 3) + B(s - 2)$$

Then

$$3 - s = As + 3A + Bs - 2B$$

So we can write two equations

$$A + B = -1, \text{and } 3A - 2B = 3$$

Solving the previous two equations we get

$$A = \frac{1}{5} \quad , \text{and } B = \frac{-6}{5}$$

By putting the values of A, and B we get

$$X(s) = \frac{1}{5}\left(\frac{1}{s - 2}\right) - \frac{6}{5}\left(\frac{1}{s + 3}\right)$$

By taking the inverse transform we get the solution of the last differential equation

$$X(t) = \frac{1}{5} e^{2t} - \frac{6}{5} e^{-3t}$$

**2-5-4 Laplace Transform of the Fractional Derivative**

We recall that the Laplace transform of $y^{(n)}$ in the integer order operation is given by

$$\mathcal{L}\{y^{(n)}\} = s^n Y(s) - s^{n-1}y(0) - s^{n-2}y'(0) \ldots - y^{(n-1)}(0)$$

$$= s^n \, Y(s) - \sum_{k=0}^{n-1} s^{n-k-1} \, y^{(K)} \, (0) \qquad (2.29)$$

Where $\mathcal{L}[y(t)] = Y(s)$

But the fractional derivative of y (t) of order α is

$$D^{\alpha} \, y(t) = D^n [D^{-u} \, y(t)] \qquad (2.30)$$

Where, n is the smallest integer greater than α >0, and u = n – α, then equation (2.30) can be written as

$$D^{\alpha} \, y(t) = D^n \left[D^{-(n-\alpha\,)} \, y(t)\right] \qquad (2.31)$$

Now, if we assume that the Laplace transform of y (t) exists, then by using equation (2.29) we have

$$\mathcal{L}\{ \, D^{\alpha} \, y(t)\} \quad = \mathcal{L}\left\{D^n \left[D^{-(n-\alpha\,)} \, y(t)\right]\right\}$$

$$= s^n \mathcal{L}\left\{D^{-(n-\alpha\,)} \, y(t)\right\} - \sum_{k=0}^{n-1} s^{n-k-1} \, D^K \, \left[D^{-(n-\alpha\,)} \, y(t)\right]_{t=0}$$

$$= s^n \left\{s^{-(n-\alpha\,)} \, Y(s)\right\} - \sum_{k=0}^{n-1} s^{n-k-1} \, D^{K-(n-\alpha\,)} \, y(0)$$

$$= s^{\alpha} \;\; Y(s) - \sum_{k=0}^{n-1} s^{n-k-1} \, D^{K-n+\alpha} \;\; y(0) \qquad (2.32)$$

In particular, if n =1 and n =2, we respectively have

$$\mathcal{L}\{ \, D^{\alpha} \, y(t)\} = \; s^{\alpha} \;\; Y(s) - D^{-(1-\alpha\,)} \, y(0), \;\; 0 < \alpha \;\; \leq 1 \qquad (2.33)$$

And

$$\mathcal{L}\{ \, D^{\alpha} \, y(t)\} = s^{\alpha} \, Y(s) - sD^{-(2-\alpha)} y(0) - D^{-(1-\alpha)} y(0) \qquad (2.34)$$

Where here in equation (2.34), $1 < \alpha \leq \; 2$

Table (2.2) gives a brief summary of some useful Laplace transform pairs. We will frequently refer to this Table. Note that the Mittag-Leffler function is very famous. (Joseph M. Kimeu (2009)).

**Table (2.2) Laplace transform pairs (Joseph M. Kimeu (2009)).**

| Y(s) | $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ |
|---|---|
| $\frac{1}{s^{\alpha}}$ | $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ |
| $\frac{1}{(s+a)^{\alpha}}$ | $\frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-at}$ |
| $\frac{1}{s^{\alpha}-a}$ | $t^{\alpha-1} E_{\alpha,\alpha}(a\, t^{\alpha})$ |
| $\frac{s^{\alpha}}{s(s^{\alpha}+a)}$ | $E_{\alpha}(-a\, t^{\alpha})$ |
| $\frac{a}{s(s^{\alpha}+a)}$ | $1 - E_{\alpha}(-a\, t^{\alpha})$ |
| $\frac{1}{s^{\alpha}(s-a)}$ | $t^{\alpha} E_{1,\alpha+1}(at)$ |
| $\frac{s^{\alpha-\beta}}{s^{\alpha}-a}$ | $t^{\beta-1} E_{\alpha,\beta}(a\, t^{\alpha})$ |
| $\frac{s^{\alpha-\beta}}{(s-a)^{\alpha}}$ | $\frac{t^{\beta-1}}{\Gamma(\beta)} F_1(\alpha, \beta, \mathrm{at})$ |
| $\frac{1}{(s-a)(s-b)}$ | $\frac{1}{a-b}(e^{at} - e^{bt}$ |

In this table, a and b $\neq$ a are real constants, $\alpha, \beta > 0$ are arbitrary.

### 2-5-5 Laplace Transform Of Caputo Derivative

The Caputo derivative with $\alpha > 0$ was defined by Caputo (1969). The Caputo definition of the fractional derivative is very useful in the time domain studies, because the initial conditions for the fractional order differential equations with the Caputo derivatives can be given in the same manner as for the ordinary differential equations with a known Physical interpretation. (J.F. Aguilar, R. Hern´andez, and D. Lieberman (2013)).

Laplace transform to Caputo fractional calculus is

$$ {}_{a}^{c}D_{t}^{\alpha}f(x) = I_{x}^{n-\alpha}\frac{d^{n}}{dx^{n}}f(x) = {}_{a}D_{t}^{-(n-\alpha)}f^{(n)}(x) \qquad (2.35) $$

$$ {}_{a}^{c}D_{t}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_{0}^{x}(x-t)^{n-\alpha-1}f^{(n)}(t)dt, \quad (n-1<\alpha\le n) \qquad (2.36) $$

Where $n \in N$

$$ \mathcal{L}\{{}_{a}^{c}D_{t}^{\alpha}f(t); s\} = s^{-(n-\alpha)}\mathcal{L}\{f^{(n)}(t)\} \qquad (2.37) $$

By using equation (2.29), we have

$$ \mathcal{L}\{{}_{a}^{c}D_{t}^{\alpha}f(t); s\} = s^{-(n-\alpha)}\left\{s^{n}f(s) - \sum_{k=0}^{n-1}s^{n-k-1}f^{(k)}(0+)\right\} \qquad (2.38) $$

$$ \mathcal{L}\{{}_{a}^{c}D_{t}^{\alpha}f(t); s\} = s^{\alpha}f(s) - \sum_{k=0}^{n-1}s^{n-k-1}f^{k}(0+); \ (n-1<\alpha\le n) \qquad (2.39) $$

where $R(s) > 0$ *and* $R(\alpha) > 0$. (Saxena and Nishimoto(2002)).

**2-6 Fractional System**

We propose a simple procedure for constructing the fractional differential equation for the fractional system. To do that, we replace the ordinary time derivative operator by the fractional one in the following way:

$$ \frac{d}{dt} \to \frac{d^{\alpha}}{dt^{\alpha}}, \qquad 0<\alpha\le 1 \qquad (2.40) $$

It can be seen that equation (2.40) is not quite right, from a physical point of view, because the time derivative operator $d/dt$ has dimension of inverse seconds $s^{-1}$, while the fractional time derivative operator $d^{\alpha}/dt^{\alpha}$ has, $s^{-\alpha}$. In order to be consistent with the time dimensionality we present the new parameter σ in the following way

$$ \left[\frac{1}{\sigma^{1-\alpha}}\frac{d^{\alpha}}{dt^{\alpha}}\right] = \frac{1}{s}. \qquad 0<\alpha\le 1 \qquad (2.41) $$

Where α is an arbitrary parameter which represents the order of the derivative. If $\alpha = 1$, equation (2.41) becomes an ordinary derivative

operator $d/d$ . In the way equation (2.41) is dimensionally consistent if and only if the new parameter σ, has dimension of time $[\sigma] = s$. (J. F. Gomez-Aguilar, J. J. Rosales-Garcia, J. J. Bernal-Alvarado, T. Cordova-Fraga, and R. Guzman-Cabrera (2012)).

Then, we have a simple procedure to construct fractional differential equations. It consists in the following, in an ordinary differential equation replace the ordinary derivative by the following fractional derivative operator

$$\frac{d}{dt} \rightarrow \frac{1}{\sigma^{1-\alpha}} \frac{d^{\alpha}}{dt^{\alpha}}, \qquad 0 < \alpha \leq 1 \tag{2.42}$$

( Moaz M Altarawneh (2015)).

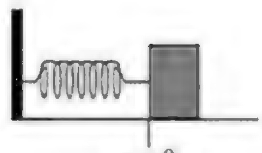
Equation (2.42) is a time derivative in the usual sense, because its dimension is $s^{-1}$.

# CHAPTER THREE

# CLASICAL TREATMENT OF DIFFERENTIAL EQUATIONS

## 3.1 Harmonic Oscillator Equation

Consider a harmonic oscillator of stiffness k attached to a block of mass m. If the block is displaced a distance x from equilibrium.( Ali M Qudah (2013)).




We neglected the friction, then Newton's second law is:

$$\sum F = ma, \qquad a = \frac{d^2x}{dt^2}$$

$$-kx(t) = m\frac{d^2x(t)}{dt^2}$$

The equation of motion can be written as

$$m\frac{d^2x(t)}{dt^2} + kx(t) = 0 \tag{3.1}$$

Also, we take as initial conditions

$$x(t = 0) = X_0$$

$$\dot{x}(t = 0) = 0$$

Now, applying the Laplace transform to equation( 3.1) we get

$$m\mathcal{L}\left\{\frac{d^2x(t)}{dt^2}\right\} + k\mathcal{L}\{x(t)\} = 0$$

From equation (2.27),The transform of second derivative is

$$\mathcal{L}\left\{\frac{d^2x(t)}{dt^2}\right\} = s^2\mathcal{L}\{x(t)\} - sx(0) - \dot{x}(0) \tag{3.2}$$

By applying the boundary conditions the equation will be

$$ms^2\mathcal{L}\{x(t)\} - msX_0 + k\mathcal{L}\{x(t)\} = 0$$

$$X(s) = \frac{msX_0}{ms^2 + k}$$

Divided by m, then let $\omega^2 = \frac{k}{m}$,

$$X(s) = X_0 \frac{s}{s^2 + \omega^2} \tag{3.3}$$

Taking the inverse Laplace transform for both sides in equation (3.3) we get

$$x(t) = X_0\mathcal{L}^{-1}\left\{\frac{s}{s^2 + \omega^2}\right\}$$

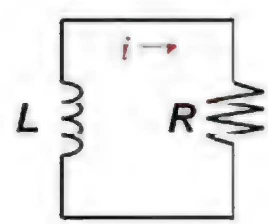
From table (2.1) the solution for this equation is given by:

$$x(t) = X_0 \cos \omega t \tag{3.4}$$

(Arfken and Weber (1985)).

**3.2 R L Circuit**

Consider an electric circuit consisting of a resistor R, inductor L, are connected in series.




The equation of motion is

$$L\frac{dI}{dt} + RI(t) = 0 \tag{3.5}$$

Now, by applying Laplace transform to equation ( 3.5) we get

$$\mathcal{L}\left\{L\frac{dI}{dt}\right\}+\ \mathcal{L}\{RI(t)\}=0$$

Using equation (2.26) we can write

$$L[s\mathcal{L}\{I(t)\}-I(0)]+R\,\mathcal{L}\{I(t)\}=0 \qquad (3.6)$$

Assume the initial condition is

$$I(t=0)=I_0$$

Equation (3.6) becomes

$$LsI(s)-LI_0\ +RI(s)=0$$

Taking the common factor I(s) we get

$$I(s)\{\ Ls+R\}=LI_0$$

$$I(s)=\frac{LI_0}{Ls+R}=\frac{LI_0}{L\left(s+\frac{R}{L}\right)}=\frac{I_0}{\left(s+\frac{R}{L}\right)} \qquad (3.7)$$

Taking the inverse Laplace transform for both sides we get

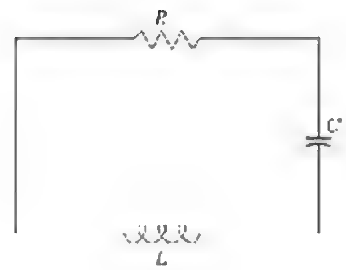
$$\mathcal{L}^{-1}\{I(s)\}=\ \mathcal{L}^{-1}\left\{\frac{I_0}{\left(s+\frac{R}{L}\right)}\right\}$$

By using table (2.1) we get

$$I(t)=\ I_0e^{-Rt/L} \qquad (3.8)$$

### 3.3 RLC Circuit

The RLC circuit is a series circuit, its elements are the resistor of resistance R, a coil of inductance L, a capacitor of capacitance C.




The sum of the potential differences around the loop must be zero (Kirchhoff`s law). This gives

$$V_L + V_R + V_C = 0$$

$$L\frac{dI(t)}{dt} + RI(t) + \frac{q(t)}{C} = 0$$

Assuming that R, L, C are known, this equation represent the differential equation with two unknowns, I and q. However $I(t) = \frac{dq(t)}{dt}$, so the equation of motion for the RLC circuit is

$$\mathrm{L}\ddot{\mathrm{q}} + \mathrm{R}\,\dot{\mathrm{q}} + \frac{\mathrm{q}}{\mathrm{C}} = 0 \qquad (3.9)$$

Which can be written as

$$\left\{\mathrm{L}\frac{\mathrm{d}^2\mathrm{q}(\mathrm{t})}{\mathrm{dt}^2}\right\} + \left\{\mathrm{R}\frac{\mathrm{dq}(\mathrm{t})}{\mathrm{dt}}\right\} + \left\{\frac{\mathrm{q}(\mathrm{t})}{\mathrm{C}}\right\} = 0 \qquad (3.10)$$

Applying Laplace transform to equation (3.10),we get

$$\mathcal{L}\left\{\mathrm{L}\frac{\mathrm{d}^2\mathrm{q}(\mathrm{t})}{\mathrm{dt}^2}\right\} + \mathcal{L}\left\{\mathrm{R}\frac{\mathrm{dq}(\mathrm{t})}{\mathrm{dt}}\right\} + \mathcal{L}\left\{\frac{\mathrm{q}(\mathrm{t})}{\mathrm{C}}\right\} = 0$$

$$\mathrm{L}\,\mathcal{L}\left\{\frac{\mathrm{d}^2\mathrm{q}(\mathrm{t})}{\mathrm{dt}^2}\right\} + \mathrm{R}\,\mathcal{L}\left\{\frac{\mathrm{dq}(\mathrm{t})}{\mathrm{dt}}\right\} + \frac{1}{\mathrm{C}}\mathcal{L}\{\mathrm{q}(\mathrm{t})\} = 0$$

Using equations (2.26), (2.27) the previous equation becomes

$$\mathrm{L}[\mathrm{s}^2\mathcal{L}\{\mathrm{q(t)}\} - \dot{\mathrm{q}}(0) - \mathrm{sq}(0)] + \mathrm{R}[\mathrm{s}\mathcal{L}\{\mathrm{q(t)}\} - \mathrm{q}(0)] + \frac{1}{\mathrm{C}}\mathcal{L}\{\mathrm{q(t)}\} = 0$$

Also, we take the initial conditions

$\mathrm{q}(\mathrm{t}=0) = q_0$

$\dot{\mathrm{q}}(\mathrm{t}=0) = 0$

$$L[s^2 q(s) - s q_0] + R[s q(s) - q_0] + \frac{1}{C} q(s) = 0$$

Where, $\mathrm{q(s)} = \mathcal{L}\{\mathrm{q(t)}\}$

$$\mathrm{q(s)} = \frac{\mathrm{sL}q_0 + \mathrm{R}q_0}{\mathrm{Ls}^2 + \mathrm{Rs} + \frac{1}{\mathrm{c}}}$$

Dividing by L, we get

$$\mathrm{q(s)} = q_0 \left[\frac{(\mathrm{sL} + \mathrm{R})/\mathrm{L}}{\left(\mathrm{Ls}^2 + \mathrm{Rs} + \frac{1}{\mathrm{c}}\right)/\mathrm{L}}\right]$$

$$q(s) = q_0 \left[\frac{s + R/L}{s^2 + \frac{R}{L}s + \frac{1}{Lc}}\right] \quad (3.11)$$

By making complete square for denominator

$$q(s) = q_0 \left[\frac{s + R/2L + R/2L}{s^2 + \frac{R}{L}s + \frac{R^2}{4L^2} + (-\frac{R^2}{4L^2} + \frac{1}{LC})}\right] \quad (3.12)$$

Let

$$\omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2} \quad > 0 \tag{3.13}$$

Then equation (3.12) becomes

$$q(s) = q_0\left(\frac{s+\frac{R}{2L}}{\left(s+\frac{R}{2L}\right)^2+\omega^2} + \frac{\frac{R}{2L}}{\left(s+\frac{R}{2L}\right)^2+\omega^2}\right) \tag{3.14}$$

By taking inverse Laplace transform for both sides of equation (3.14), we get

$$\mathcal{L}^{-1}\{q(s)\} = q_0\left(\mathcal{L}^{-1}\left\{\frac{s+\frac{R}{2L}}{\left(s+\frac{R}{2L}\right)^2+\omega^2}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{R}{2L}\times\frac{\omega}{\omega}}{\left(s+\frac{R}{2L}\right)^2+\omega^2}\right\}\right)$$

From table (2.1) we have,

$$\mathrm{q(t)} = q_0\,\mathrm{e}^{-\mathrm{t}R/2L}\cos(\omega t) + \frac{R}{2L\omega}q_0\,\mathrm{e}^{-\mathrm{t}R/2L}\sin(\omega t)$$

$$\mathrm{q(t)} = q_0\,\mathrm{e}^{-\mathrm{t}R/2L}\left[\cos(\omega t) + \frac{R}{2L\omega}\sin(\omega t)\right] \tag{3.15}$$

# CHAPTER FOUR

# FRACTIONAL TREATMENT

The fractional calculus plays very important role in the investigations to be more accurate (Agrawal, 2001). In this chapter we will depend on the fractional calculus and use the fractional action to obtain the fractional Solution for the harmonic oscillator equation, RL circuit equation, RLC circuit equation.

## 4.1 Fractional Solution For The Harmonic Oscillator Equation

In order to find the fractional solution for the ordinary differential equation for the harmonic oscillator equation (3.1) we propose to replace the ordinary time derivative operator by the fractional time derivative operator. (J. F. Gomez-Aguilar, J. J. Rosales-Garcia, J. J. Bernal-Alvarado, T. Cordova-Fraga, and R. Guzman-Cabrera (2012))

$$\frac{d}{dt} \rightarrow \frac{1}{\sigma^{1-\alpha}} \frac{d^{\alpha}}{dt^{\alpha}}, \qquad n-1 < \alpha \leq n \tag{4.1}$$

$$\frac{d^2}{dt^2} \rightarrow \frac{1}{\sigma^{2(1-\alpha)}} \frac{d^{2\alpha}}{dt^{2\alpha}}, \qquad n-1 < 2\alpha \leq n \tag{4.2}$$

Where $\alpha$ is an arbitrary parameter that represents the order of the time derivative . The parameter $\sigma$ is a new parameter introduced for the differential equation whose independent variable has the dimension of time.
Equation (3.1) can be written as

$$\ddot{x} + {\omega_0}^2 \, x(t) = 0 \tag{4.3}$$

Where ${\omega_0}^2 = \frac{k}{m}$

Applying equation (4.2) to equation (4.3) we get

$$\frac{1}{\sigma^{2(1-\alpha)}} \frac{d^{2\alpha}}{dt^{2\alpha}} x(t) + {\omega_0}^2 x(t) = 0$$

Which can be written as

$$\frac{1}{\sigma^{2(1-\alpha)}}\, D^{2\alpha}\, x(t) + {\omega_0}^2 x(t) = 0$$

Multiplying the previous equation by $\sigma^{2(1-\alpha)}$ we get

$$D^{2\alpha}\, x(t) + {\omega_0}^2\, \sigma^{2(1-\alpha)} x(t) = 0 \tag{4.4}$$

Let $\omega^2 = {\omega_0}^2\, \sigma^{2(1-\alpha)}$

Then equation (4.4) becomes

$$D^{2\alpha}\, x(t) + \omega^2 x(t) = 0 \tag{4.5}$$

Applying Laplace transform for the fractional derivative equation (2.32) to equation (4.5) we get

$$\mathcal{L}\{D^{2\alpha} x(t)\} + \omega^2 \mathcal{L}\{x(t)\} = 0$$

$$s^{2\alpha}\, X(s) - \sum_{k=0}^{n-1} s^{n-k-1}\, D^{k-(n-2\alpha)} x(t)\Big|_{t=0} \quad + \omega^2 X(s) = 0 \tag{4.6}$$

Where $\mathcal{L}\{x(t)\} = X(s), n = 2$

Equation (4.6) can be written as

$$s^{2\alpha}\, X(s) - \left[ sD^{2\alpha-2} x(t)\Big|_{t=0} \quad + D^{2\alpha-1} x(t)\Big|_{t=0} \quad \right] + \omega^2 X(s) = 0 \tag{4.7}$$

Taking the common factor X(s) we get

$$X(s)[s^{2\alpha} + \omega^2] = \left[ sD^{2\alpha-2} x(t)\Big|_{t=0} \quad + D^{2\alpha-1} x(t)\Big|_{t=0} \quad \right]$$

$$X(s) = \frac{[sD^{2\alpha-2} + D^{2\alpha-1}] x(t)\Big|_{t=0}}{s^{2\alpha} + \omega^2} \tag{4.8}$$

Applying the initial conditions

For α=1 then,

$$D^{2\alpha-2}x(t)\Big|_{t=0} \quad = x(0) = X_0$$

And

$$D^{2\alpha-1}x(t)\Big|_{t=0} \quad = Dx(0) = \dot{x}\,(0) = 0$$

Equation (4.8) will be

$$X(s) = \frac{sX_0}{s^{2\alpha}+\omega^2} \;=\; X_0\,\frac{s}{s^{2\alpha}+\omega^2} \tag{4.9}$$

Taking the inverse Laplace transform for both sides and using table (2.2) we get

$$X(t) = \; X_0\mathcal{L}^{-1}\left[\frac{s}{s^{2\alpha}+\omega^2}\right] \tag{4.10}$$

But

$$\mathcal{L}^{-1}\left[\frac{s}{s^{2\alpha}+\omega^2}\right] = \mathcal{L}^{-1}\left[\frac{s^{2\alpha-(2\alpha-1)}}{s^{2\alpha}+\omega^2}\right] = \mathcal{L}^{-1}\left[\frac{s^{2\alpha-\beta)}}{s^{2\alpha}+\omega^2}\right]$$

Where $\beta = 2\alpha - 1$, then we have

$$\mathcal{L}^{-1}\left[\frac{s}{s^{2\alpha}+\omega^2}\right] = t^{\beta-1}E_{2\alpha,\beta}(-\omega^2 t^{2\alpha}\,) = t^{2\alpha-2}E_{2\alpha,2\alpha-1}(-\omega^2 t^{2\alpha}\,) \tag{4.11}$$

So equation (4.10) becomes

$$X(t) = X_0 t^{2\alpha-2}E_{2\alpha,2\alpha-1}(-\omega^2 t^{2\alpha}\,) \tag{4.12}$$

$$X(t) = X_0 t^{2\alpha-2}\sum_{k=0}^{\infty}\frac{(-\omega^2 t^{2\alpha}\,)^k}{\Gamma(2\alpha k + 2\alpha - 1)} \tag{4.13}$$

For a special case $\alpha = 1, \omega = \omega_0$, equation (4.13) becomes

$$X(t) = X_0 \sum_{k=0}^{\infty} \frac{(-1)^k (\omega_0 t)^{2k}}{\Gamma(2k+1)} = X_0 \sum_{k=0}^{\infty} \frac{(-1)^k (\omega_0 t)^{2k}}{(2k)!} \tag{4.14}$$

So we can write the final solution as

$$x(t) = X_0 \cos \omega t \tag{4.15}$$

In fact, this result is in full agreement with that obtained in the classical case equation (3.4)

**4.2 Fractional Solution For The RL Circuit Equation**

The equation of the RL circuit (3.5) was

$$\mathrm{L}\dot{\mathrm{I}} + \mathrm{RI(t)} = 0$$

Applying equation (4.1) to the previous equation we get

$$L \frac{1}{\sigma^{(1-\alpha)}} \frac{d^{\alpha}}{dt^{\alpha}} I(t) + R\, I(t) = 0 \tag{4.16}$$

Which can be also written as

$$L \frac{1}{\sigma^{(1-\alpha)}} D^{\alpha} I(t) + R\, I(t) = 0 \tag{4.17}$$

Multiplying the previous equation by $\sigma^{(1-\alpha)}$ and dividing it by L we get

$$D^{\alpha} I(t) + \frac{R}{L} \sigma^{(1-\alpha)} I(t) = 0 \tag{4.18}$$

By applying Laplace transform for the fractional derivative equation (2.32) to equation (4.18), we get

$$\mathcal{L} \{D^{\alpha} I(t)\} + \frac{R}{L} \sigma^{(1-\alpha)} \mathcal{L} \{I(t)\} = 0 \tag{4.19}$$

$$s^{\alpha} I(s) - \sum_{k=0}^{n-1} s^{n-k-1} D^{k-(n-\alpha)} I(t)\big|_{t=0} + \frac{R}{L} \sigma^{(1-\alpha)} I(s) = 0 \tag{4.20}$$

Where$\mathcal{L}$ $\{I(t)\} = I(s)$, n=1

Equation (4.20) can be written as

$$s^{\alpha}\, I(s) - D^{\alpha-1} I(t)\big|_{t=0} \quad + \acute{R}\, I(s) = 0 \tag{4.21}$$

Where $\acute{R} = \frac{R}{L}\ \sigma^{(1-\alpha)}$

Applying the initial condition
For a special case

$(\alpha = 1)$, then $D^{\alpha-1} I(t)\big|_{t=0} \quad = D^{0} I(t)\big|_{t=0} \quad = I(0) = I_0$

Equation (4.21) becomes

$$s^{\alpha}\, I(s) - I_0 + \acute{R}\, I(s) = 0 \tag{4.22}$$

Taking the common factor I(s) we get

$$I(s)\left[s^{\alpha} + \acute{R}\,\right] = I_0 \tag{4.23}$$

$$I(s) = \frac{I_0}{\left[s^{\alpha} + \acute{R}\,\right]} \tag{4.24}$$

Taking the inverse Laplace transform for both sides of equation (4.24) and using table (2.2) we get

$$I(t) = I_0 t^{\alpha-1} E_{\alpha,\alpha}\left(-\acute{R}\, t^{\alpha}\right) \tag{4.25}$$

$$I(t) = I_0 E_{\alpha,\alpha}\left(-\frac{R}{L}\ \sigma^{(1-\alpha)} t^{\alpha}\right)\ t^{\alpha-1} \tag{4.26}$$

For a special case if $\alpha = 1$, we get

$$I(t) = I_0 E_{1,1}\left(-\frac{R}{L}\ t\right) \tag{4.27}$$

$$I(t) = I_0 \sum_{n=0}^{\infty} \frac{\left(\frac{-Rt}{L}\right)^{n}}{\Gamma(n+1)} \tag{4.28}$$

$$I(t) = I_0 \sum_{n=0}^{\infty} \frac{\left(\frac{-Rt}{L}\right)^n}{n!} \tag{4.29}$$

Equation (4.29) can be written as

$$I(t) = I_0 e^{-Rt/L} \tag{4.30}$$

Again , this result is in full agreement with that obtained in the classical case equation (3.8).

**4.3 Fractional Solution For The RLC Circuit Equation**

The equation of the RLC circuit is

$$L\ddot{q} + R\dot{q} + \frac{q(t)}{C} = 0$$

Applying equation (4.1) and equation (4.2) to the previous equation we get

$$L \frac{1}{\sigma^{2(1-\alpha)}} \frac{d^{2\alpha}}{dt^{2\alpha}} q(t) + R \frac{1}{\sigma^{(1-\alpha)}} \frac{d^{\alpha}}{dt^{\alpha}} q(t) + \frac{q(t)}{C} = 0 \tag{4.31}$$

Which can be written as

$$L \frac{1}{\sigma^{2(1-\alpha)}} D^{2\alpha} q(t) + R \frac{1}{\sigma^{(1-\alpha)}} D^{\alpha} q(t) + \frac{q(t)}{C} = 0 \tag{4.32}$$

Multiplying equation (4.32) by $\sigma^{2(1-\alpha)}$ and dividing it by L we get

$$D^{2\alpha} q(t) + \frac{R}{L} \frac{\sigma^{2(1-\alpha)}}{\sigma^{(1-\alpha)}} D^{\alpha} q(t) + \sigma^{2(1-\alpha)} \frac{q(t)}{LC} = 0 \tag{4.33}$$

Which can be written as

$$D^{2\alpha} q(t) + \frac{R}{L} \sigma^{(1-\alpha)} D^{\alpha} q(t) + \sigma^{2(1-\alpha)} \frac{q(t)}{LC} = 0 \tag{4.34}$$

Let

$$\omega_1^2 = \frac{R}{L} \sigma^{(1-\alpha)} \tag{4.35}$$

And

$$\omega_2^2 = \frac{\sigma^{2(1-\alpha)}}{LC} \tag{4.36}$$

Then equation (4.34) becomes

$$D^{2\alpha}\, q(t) + \omega_1^2\, D^{\alpha}\, q(t) + \omega_2^2\, q(t) = 0 \tag{4.37}$$

Applying Laplace transform for the fractional derivative equation (2.32) to equation (4.37) we get

$$\mathcal{L}\{D^{2\alpha}\, q(t)\} + \omega_1^2 \mathcal{L}\{D^{\alpha}\, q(t)\} + \omega_2^2 \mathcal{L}\{q(t)\} = 0$$

$$s^{2\alpha}\, q(s) - \sum_{k=0}^{n-1} s^{n-k-1}\, D^{k-(n-2\alpha)} q(t)\big|_{t=0} \quad + \omega_1^2\Big[s^{\alpha}\, q(s) - \sum_{k=0}^{n-1} s^{n-k-1}\, D^{k-(n-\alpha)} q(t)\big|_{t=0} \quad \Big] + \omega_2^2 q(s) = 0 \tag{4.38}$$

Then

$$s^{2\alpha}\, q(s) - \sum_{k=0}^{1} s^{2-k-1}\, D^{k-(2-2\alpha)} q(t)\big|_{t=0} \quad + \omega_1^2\Big[s^{\alpha}\, q(s) - \sum_{k=0}^{0} s^{1-k-1}\, D^{k-(1-\alpha)} q(t)\big|_{t=0} \quad \Big] + \omega_2^2 q(s) = 0 \tag{4.39}$$

So we can write

$$s^{2\alpha}\, q(s) - \Big[sD^{(2\alpha-2)} q(t)\big|_{t=0} \quad + D^{(2\alpha-1)} q(t)\big|_{t=0} \quad \Big] + \omega_1^2\Big[s^{\alpha}\, q(s) - D^{(\alpha-1)} q(t)\big|_{t=0} \quad \Big] + \omega_2^2 q(s) = 0 \tag{4.40}$$

Then equation (4.40) can be written as

$$s^{2\alpha}\, q(s) - sD^{(2\alpha-2)} q(t)\big|_{t=0} \quad - D^{(2\alpha-1)} q(t)\big|_{t=0} \quad + \omega_1^2 s^{\alpha}\, q(s) - \omega_1^2 D^{(\alpha-1)} q(t)\big|_{t=0} \quad + \omega_2^2 q(s) = 0 \tag{4.41}$$

Taking the common factor q(s) we get

$$q(s)[s^{2\alpha} + \omega_1^2 s^{\alpha} + \omega_2^2] = sD^{(2\alpha-2)} q(t)\big|_{t=0} \quad + D^{(2\alpha-1)} q(t)\big|_{t=0} \quad + \omega_1^2 D^{(\alpha-1)} q(t)\big|_{t=0}$$

Then

$$q(s) = \frac{sD^{(2\alpha-2)}q(t)\big|_{t=0} \quad +D^{(2\alpha-1)}q(t)\big|_{t=0} \quad +\omega_1^2 D^{(\alpha-1)}q(t)\big|_{t=0}}{[s^{2\alpha}+\omega_1^2 s^{\alpha}+\omega_2^2]} \tag{4.42}$$

Applying the initial conditions

For $\alpha = 1$ then,

$$sD^{(2\alpha-2)}q(t)\big|_{t=0} \quad = sq(0) = sq_0$$

And

$$D^{(2\alpha-1)}q(t)\big|_{t=0} \quad = Dq(0) = \dot{q}\ (0) = 0$$

And

$$\omega_1^2 D^{(\alpha-1)}q(t)\big|_{t=0} \quad = \omega_1^2 q(0) = \omega_1^2 q_0$$

Then equation (4.42) can be written as

$$q(s) = \frac{sq_0 + \omega_1^2 q_0}{s^{2\alpha} + \omega_1^2 s^{\alpha} + \omega_2^2} \tag{4.43}$$

$$q(s) = q_0\ \frac{s + \omega_1^2}{s^{2\alpha} + \omega_1^2 s^{\alpha} + \omega_2^2} \tag{4.44}$$

By making complete square for denominator in equation (4.44) we get

$$q(s) = q_0\ \frac{s + \omega_{1/2}^2 + \omega_{1/2}^2}{\left[s^{2\alpha} + \omega_1^2 s^{\alpha} + \frac{\omega_1^4}{4}\right] + \left[\omega_2^2 - \frac{\omega_1^4}{4}\right]} \tag{4.45}$$

Now let

$$C_0^2 = \omega_2^2 - \frac{\omega_1^4}{4} \tag{4.46}$$

Equation (4.45) becomes

$$q(s) = q_0\ \frac{s + \omega_{1/2}^2 + \omega_{1/2}^2}{\left(s^{\alpha} + \frac{\omega_1^2}{2}\right)^2 + C_0^2} \tag{4.47}$$

Which can be written as

$$q(s) = q_0 \frac{s + \omega_{1/2}^2}{\left(s^\alpha + \frac{\omega_1^2}{2}\right)^2 + C_0^2} + q_0 \frac{\omega_{1/2}^2}{\left(s^\alpha + \frac{\omega_1^2}{2}\right)^2 + C_0^2} \tag{4.48}$$

Taking the inverse Laplace transform for both sides and using table (2.1) we get

$$q(t) = q_0 e^{\frac{-\omega_1^2 t}{2}} \cos C_0 t + \mathcal{L}^{-1}\left[q_0 \frac{\omega_{1/2}^2}{\left(s^\alpha + \frac{\omega_1^2}{2}\right)^2 + C_0^2}\right] \tag{4.49}$$

To calculate the second term we multiply it by $\frac{C_0}{C_0}$ then we get

$$\mathcal{L}^{-1}\left[q_0 \frac{\omega_{1/2}^2}{\left(s^\alpha + \frac{\omega_1^2}{2}\right)^2 + C_0^2}\right] = \frac{\omega_1^2}{2C_0} \mathcal{L}^{-1}\left[q_0 \frac{C_0}{\left(s^\alpha + \frac{\omega_1^2}{2}\right)^2 + C_0^2}\right] \tag{4.50}$$

Using table (2.1) we get

$$\mathcal{L}^{-1}\left[q_0 \frac{\frac{\omega_1^2}{2}}{\left(s^\alpha + \frac{\omega_1^2}{2}\right)^2 + C_0^2}\right] = \frac{\omega_1^2}{2C_0} q_0 e^{\frac{-\omega_1^2 t}{2}} \sin C_0 t \tag{4.51}$$

Substituting equation (4.51) in equation (4.49) we get

$$q(t) = q_0 e^{\frac{-\omega_1^2 t}{2}} \left[\cos C_0 t + \frac{\omega_1^2}{2C_0} \sin C_0 t\right] \tag{4.52}$$

Now by substituting $\omega_1^2, \omega_2^2,\ C_0^2,\ C_0$ from equations (4.35), (4.36), and (4.46) we get

$$q(t) = q_0 e^{\frac{-Rt}{2L}\sigma^{(1-\alpha)}}\left[\cos\sqrt{\frac{\sigma^{2(1-\alpha)}}{LC}-\frac{R^2\sigma^{2(1-\alpha)}}{4L^2}}\,t + \frac{R\sigma^{(1-\alpha)}}{2L\sqrt{\frac{\sigma^{2(1-\alpha)}}{LC}-\frac{R^2\sigma^{2(1-\alpha)}}{4L^2}}}\sin\sqrt{\frac{\sigma^{2(1-\alpha)}}{LC}-\frac{R^2\sigma^{2(1-\alpha)}}{4L^2}}\,t\right] \quad (4.53)$$

Now for a special case if $\alpha = 1$, the previous equation becomes

$$q(t) = q_0 e^{\frac{-R}{2L}}\left[\cos\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\,t + \frac{R}{2L\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}\sin\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\,t\right] \quad (4.54)$$

But $\frac{1}{LC}-\frac{R^2}{4L^2}=\omega^2$ in the classical solution, so equation (4.54) becomes

$$q(t) = q_0\, e^{-Rt/2L}\left[\cos(\omega t) + \frac{R}{2L\omega}\sin(\omega t)\right] \quad (4.55)$$

This result is in full agreement with that obtained in the classical case equation (3.15)

## 4.4 Conclusion

Fractional calculus is a more generalized form of calculus. We investigated about fractional calculus including some of useful mathematical functions. Also we reviewed fractional Laplace transform, which is applied in this thesis to some fractional differential equations

including conservative systems such as, harmonic oscillator equation, and for non-conservative systems such as, RL, and RLC circuits.

We found that the fractional solutions are in full agreement with that obtained in the classical cases when $\alpha = 1$, where $\alpha$ is an arbitrary parameter which represents the order of the derivative.

### 4.5 Open problem

Equations of motion can be rewritten using the fractional calculus for the U-tube equation

$$l\ddot{x} + 2gx = 0$$

Where $l$ is the length of the U-tube.

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